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Nonlinear Functionals of Multi-D Discrete Velocity Boltzmann Equations

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In this paper, we study nonlinear functionals measuring potential interactions and L^1-distance between two mild solutions for the multi-dimensional discrete velocity Boltzmann equations when the initial data are a small perturbation of a vacuum. We employ Bony's dispersion estimates to show that these functionals satisfy Lyapunov type estimates which are useful for the study of time-asymptotics and L^1-stability of mild solutions.

KEY WORDS: Discrete velocity Boltzmann equations; L^1-stability; nonlinear functionals.

1. INTRODUCTION

The purpose of this paper is to study nonlinear functionals of the multi-dimensional discrete velocity Boltzmann equations:

$$\partial_t f_i(x,t) + v_i \cdot \nabla_x f_i(x,t) = Q_i(f,f)(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad 1 \leqslant i \leqslant N, \tag{1.1}$$

where f_i is the density of particles with velocity $v_i = (v_i^1, \ldots, v_i^n)$ and the system is assumed to be strictly hyperbolic in the sense that all characteristic velocities are distinct. Moreover, we assume that the collision operator $Q_i(f,f)$ satisfies the transversality assumption:

$$Q_i(f,f) \equiv \sum_{1 \leqslant j,k \leqslant N} B_i^{jk} f_j f_k, \qquad B_i^{jk} = 0, \text{ if } j = k \qquad \text{and}$$
$$0 < \max_{i,j,k} |B_i^{jk}| =: B_* < \infty. \tag{1.2}$$

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The above transversality assumption (1.2) of $Q_i(f, f)$ is natural in the sense that if pre-collision velocities are equal, then post-collision velocities should be equal to the pre-collision velocities, hence the collisions between particles with the same velocity will not contribute to $Q_i(f, f)$. The system (1.1) includes standard multi-dimensional discrete velocity Boltzmann models such as the three-dimensional Broadwell model.[7]

Discrete velocity Boltzmann equations were originally introduced with the idea of getting simpler models for some fundamental questions such as shock wave structures,[7] boundary layers, approximation schemes for the full Boltzmann equation,[21, 29] analytical solutions etc. However, it does not seem to be true that discrete velocity Boltzmann models are simpler than the Boltzmann equation. The definition of solutions in the mild sense can be stated as follows.

Definition 1.1. Let $f = (f_1,..., f_N) \in C([0, T]; (L^1(\mathbb{R}^n) \cap L^\infty_+(\mathbb{R}^n))^N)$ be the mild solution of (1.1) with given nonnegative initial data $f_0 \in (L^1(\mathbb{R}^n) \cap (L^\infty_+(\mathbb{R}^n))^N$ if and only if for all $t \in [0, T]$ and a.e $x \in \mathbb{R}^n$, $f(x, t)$ satisfies the following integral equation:

$$f_i(x, t) = f_{0i}(x - tv_i) + \int_0^t Q_i(f, f)(x - (t-s)\, v_i, s)\, ds, \qquad i = 1,..., N.$$

The global existence and uniqueness of mild solutions for the discrete velocity Boltzmann models (1.1) was first obtained by Nishida and Mimura[25] for the one-dimensional Broadwell model with small L^1-data, and this small data existence theory was generalized by Tartar,[27, 28] Beale,[1, 2] and Bony[3] to the one-dimensional discrete velocity Boltzmann models with large L^1-data. For further references, we refer to the survey article by Illner and Platkowski.[19] Throughout our paper, we use a simplified notation for the L^p-norm:

$$\|f(t)\|_{L^p} \equiv \sum_{i=1}^{N} \|f_i(\cdot, t)\|_{L^p(\mathbb{R}^n)}.$$

Since the L^1-norm of f in x at time t is equal to the total mass of particles at time t and is invariant in time t, the L^1-norm is expected to be a natural norm for stability analysis. In fact, L^1-stability has been quite successful in one-dimensional hyperbolic systems of conservation laws.[6, 24] In the context of collisional kinetic equations, L^1-stability for one-dimensional discrete velocity Boltzmann models with only cross-interaction terms was first obtained by Tartar,[28] and Ha and Tzavaras[16] proved L^1-stability for some subclass of discrete velocity Boltzmann models such as Beale's models[2] containing self-interaction terms using a robust Lyapunov functional approach. Moreover, this Lyapunov functional approach for L^1-stability has been applied to several one-dimensional kinetic models with collision terms such as coagulation models[11] and a one-dimensional Boltzmann-type equation with inelastic collisions.[14]

On the other hand, the global existence and uniqueness of mild solutions for multi-dimensional discrete velocity Boltzmann models (1.1) has been extensively studied in refs. 4, 17, 18, 22, and 23. Recently in ref. 15, the second author succeeded in proving the L^1-stability of some class of mild solutions for (1.1):

$$\|f(t)-\bar{f}(t)\|_{L^1} \leqslant G\,\|f_0-\bar{f}_0\|_{L^1}. \tag{1.3}$$

where G is a positive constant independent of time t.

In this paper, we consider initial data which are a small perturbation of a trivial equilibrium state "*vacuum*." Since we are assuming strict hyperbolicity **(M1)** below, initial perturbations will be propagated into the vacuum state along the characteristics with different speeds "*(dispersion)*," hence time-asymptotically the total density is expected to decay to the vacuum state pointwise, and the total collisions will be decreasing so that mild solutions will tend to the collisionless flow time-asymptotically. In fact, these dispersive phenomena were first noticed by Tartar[27, 28] and were employed for the study of the global existence of mild solutions to the hard-sphere model of the Boltzmann equation in ref. 20 when a small amount of gases expand into a vacuum.

The main novelty of this paper is to quantify the possible decay of the potential interactions between particles with different velocities by devising Lyapunov functionals.

Below we list the main assumptions **(M)** in this paper.

For a given i, we set

$$\Lambda_i \equiv \Lambda_i^+ \cup \Lambda_i^-, \qquad \Lambda_i^+ \equiv \{(j,k)\colon B_i^{jk} > 0\}, \qquad \Lambda_i^- \equiv \{(j,k)\colon B_i^{jk} < 0\},$$

$$N_*^+ \equiv \max_{i=1}^N |\Lambda_i^+|, \qquad N_*^- \equiv \max_{i=1}^N |\Lambda_i^+|, \qquad N_* \equiv \max_{i=1}^N |\Lambda_i|.$$

(M1) Strict hyperbolicity: $(v_i \neq v_j \text{ if } i \neq j)\min_{i\neq j} |v_i - v_j| \geqslant v_* > 0$,

(M2) Transversality of $Q_i(f,f)$: $(B_i^{jk} \neq 0 \Rightarrow j \neq k)$,

(M3) Smallness of initial data: $B_* N^2 N_* \,\|f_0\|_{\mathscr{E}} \leqslant \eta \ll 1$,

where the norm $\|\cdot\|_{\mathscr{E}}$ will be defined in Section 2.

Remark 1.1. (1) We do not use any conservation laws and an entropy condition for the L^1-stability analysis. However, the smallness assumption on the initial data is needed for the existence and stability analysis. Moreover, the smallness of the initial data in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is not sufficient to guarantee the uniform boundedness of mild solutions as shown in ref. 18.

(2) For a related Lyapunov functional approach, we refer to refs. 8–10, where the Bony Lyapunov functional[5] was generalized to the Boltzmann equation with some truncated collision kernels.

In our paper, by *Bony solution* we denote the mild solutions in ref. 4; the main purpose of this paper is to devise explicit nonlinear functionals measuring potential interactions and L^1-distance, which satisfy Lyapunov-type estimates.

More precisely, in Section 3, we will define an interaction potential $\mathscr{D}$ measuring all possible potential interactions between particles with different velocities:

$$\mathscr{D}(t) \equiv \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} f_i(x+tv_i, t) \left[\int_0^\infty f_j(x+v_i t + \tau n(v_i, v_j), t)\, d\tau \right] dx,$$

where the summation is over all pairs $(i, j), i \neq j$, and $n(v_i, v_j)$ is the unit vector in the direction of $v_i - v_j$:

$$n(v_i, v_j) = \frac{v_i - v_j}{|v_i - v_j|}, \qquad \text{where} \quad i \neq j.$$

Similarly, we will define a nonlinear functional $\mathscr{H}$ which is equivalent to L^1-distance:

$$\mathscr{H}(t) \equiv \sum_{i=1}^{N} \int_{\mathbb{R}^n} |f_i - \bar{f}_i| \, (x+tv_i, t) \times \left[1 + K \int_0^\infty \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} (f_j + \bar{f}_j)(x+tv_i + \tau n(v_i, v_j), t)\, d\tau \right] dx,$$

where K is a positive constant determined later. According to the estimates in Proposition 2.1 of Section 2, we have

$$0 \leqslant \int_0^\infty (f_j + \bar{f}_j)(x+tv_i+\tau n(v_i, v_j), t)\, d\tau \leqslant (1+\mathcal{O}(\eta))(\|f_0\|_{\mathscr{E}} + \|\bar{f}_0\|_{\mathscr{E}}) \ll 1,$$

hence it is easy to see that $\mathscr{H}$ is equivalent to the L^1-distance:

$$\|f(t) - \bar{f}(t)\|_{L^1} \leqslant \mathscr{H}(t) \leqslant C_0 \, \|f(t) - \bar{f}(t)\|_{L^1},$$

where C_0 is some positive constant independent of time t. The main results of this paper are the following two theorems.

Theorem 1.1. Suppose that the main assumptions **(M)** hold, and let f be the Bony solution of (1.1) corresponding to initial data f_0. Then the interaction potential $\mathscr{D}$ satisfies a Lyapunov-type estimate:

$$\mathscr{D}(t)+\frac{v_*}{2}\sum_{1\leqslant i\neq j\leqslant N}\int_0^t\int_{\mathbb{R}^n}(f_i f_j)(x,s)\,dx\,ds\leqslant\mathscr{D}(0).$$

Theorem 1.2. Suppose that the main assumptions **(M)** hold, and let f and $\bar{f}$ be Bony solutions of (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively. Then we have a quasi-Lyapunov-type estimate:

$$\mathscr{H}(t)+C_1\sum_{1\leqslant i\neq j\leqslant N}\int_0^t\int_{\mathbb{R}^n}[|f_i-\bar{f}_i|\,(f_j+\bar{f}_j)](x,s)\,dx\,ds\leqslant\bar{C}_1\mathscr{H}(0),$$

where C_1 and $\bar{C}_1$ are positive constants independent of t.

Remark 1.2. Notice that Theorem 1.2 also implies L^1-stability (1.3).

The rest of this paper is organized as follows. In Section 2, we review the basics of the Bony theory for the multi-dimensional discrete velocity Boltzmann equations. Finally, in Section 3, we explicitly construct nonlinear functionals and estimate their time-evolutions.

2. PRELIMINARIES

In this section, we briefly review basic estimates of the multi-dimensional discrete velocity Boltzmann model (1.1). The standard multi-dimensional discrete velocity Boltzmann models are:

$$\partial_t f_i+v_i\cdot\nabla_x f_i=\sum_{j,k,l}(A_{ij}^{kl}f_k f_l-A_{kl}^{ij}f_i f_j),\qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+,\quad (2.4)$$

where the collision coefficients A_{ij}^{kl} satisfy symmetry and micro-reversibility conditions:

$$A_{ij}^{kl}=A_{ij}^{lk}=A_{ji}^{kl},\qquad A_{ij}^{kl}=A_{kl}^{ij}.$$

The pre-collision velocities v_i, v_j and post-collision velocities v_k, v_l satisfy microscopic conservation laws of mass, momentum, and energy:

$$v_i+v_j=v_k+v_l,\qquad |v_i|^2+|v_j|^2=|v_k|^2+|v_l|^2.$$

Next, we briefly review the properties of the collision operator $Q_i(f, f)$:

$$Q_i(f, f) \equiv \sum_{j,k,l} A_{ij}^{kl}(f_k f_l - f_i f_j).$$

Let ϕ: $\mathbb{R}^m \to \mathbb{R}$ be any measurable function. Then we have

$$\partial_t \left(\sum_i \phi(v_i) f_i \right) + \text{div}_x \left(\sum_i v_i \phi(v_i) f_i \right) = \sum_{i,j,k,l} \phi(v_i) A_{ij}^{kl}(f_k f_l - f_i f_j).$$

The R.H.S. of the above equation can be rearranged:

$$\text{R.H.S.} = \tfrac{1}{2} \sum_{i,j,k,l} A_{ij}^{kl}(\phi(v_k) + \phi(v_l) - \phi(v_i) - \phi(v_j)) f_k f_l.$$

For the choice of $\phi(v_i) = 1, v_i^k, |v_i|^2$, we can see that the system (2.4) satisfies the conservation of mass, momentum, and energy:

$$\frac{d}{dt}\left(\sum_{i=1}^N \int_{\mathbb{R}^n} f_i(x, t)\, dx, \sum_{i=1}^N \int_{\mathbb{R}^n} v_i f_i(x, t)\, dx, \sum_{i=1}^N \int_{\mathbb{R}^n} |v_i|^2 f_i(x, t)\, dx \right) = 0.$$

On the other hand, we multiply (2.4) by $1 + \log f_i$ to get an entropy inequality:

$$\partial_t \left(\sum_i f_i \log f_i \right) + \text{div}_x \left(\sum_i v_i f_i \log f_i \right)$$
$$= -\frac{1}{4} \sum_{i,j,k,l} A_{ij}^{kl} \log \left(\frac{f_k f_l}{f_i f_j} \right) (f_k f_l - f_i f_j) \leqslant 0.$$

Below we review estimates for the Bony solutions in refs. 4 and 15. We first define a characteristic vector $V_i \in \mathbb{R}^n \times \mathbb{R}_+$ generated by a velocity $v_i \in \mathbb{R}^n$:

$$V_i = (v_i, 1) \in \mathbb{R}^n \times \mathbb{R}_+, \qquad 1 \leqslant i \leqslant N.$$

Definition 2.1.[(4, 15)]

(1) Π is a characteristic p-plane in $\mathbb{R}^n \times \mathbb{R}_+$, $1 \leqslant p \leqslant n+1$, if and only if it is an affine p-plane spanned by exactly p linearly independent characteristic vectors $V_{i_1},..., V_{i_p}$.

(2) π is a p-plane of trace type in $\mathbb{R}^n$, $0 \leqslant p \leqslant n$, if and only if it is the intersection of some $(p+1)$-characteristic plane Π and a hyperplane $\mathbb{R}^n \times \{t = T\}$, for some $T \geqslant 0$.

(3) For a given p-plane of trace type π, $J(\pi)$ denotes the set of all indices i such that V_i is parallel to Π, for some $(p+1)$-characteristic plane Π with $\pi = \Pi \cap (\mathbb{R}^n \times \{t = T\})$ for $T \geqslant 0$.

For a measurable function $f = (f_i)_{i=1}^N$, we define auxiliary functions: For $T > 0$, $0 \leqslant p \leqslant n$,

- $\delta_p(f(T)) \equiv \sup_i \operatorname{esssup}_{i \in J(\pi)} \{\int_\pi f_i(x, T)\, d^p x \mid \pi$ is a p-plane of trace type$\}$,
- $M_{p+1}(f(T)) \equiv \sup_{i \neq j, k} \sup_\Pi \{\int_{\Pi \cap (\mathbb{R}^n \times [0, T])} (|B_k^{ij}|\, f_i f_j)(P)\, d^p P \mid \Pi$ is a characteristic $(p+1)$-plane$\}$,
- $M_{p+1}(f(T), \bar{f}(T)) \equiv \sup_{i \neq j, k} \sup_\Pi \{\int_{\Pi \cap (\mathbb{R}^n \times [0, T])} (|B_k^{ij}|\, |f_i - \bar{f}_i|\, (f_j + \bar{f}_j))(P)\, d^{p+1} P \mid \Pi$ is a characteristic $(p+1)$-plane$\}$,

Now we define the Bony norm and space as follows. For any measurable function g,

$$\|g\|_{\mathscr{E}} \equiv \max_{0 \leqslant p \leqslant n} \delta_p(g), \qquad \mathbf{E} \equiv \{g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : \|g\|_{\mathscr{E}} < \infty\}.$$

Notice that $\delta_0(f(t))$ and $\delta_n(f(t))$ are equivalent to $\|f(t)\|_{L^\infty}$ and $\|f(t)\|_{L^1}$ respectively, and the space $\mathbf{E}$ is a closed subspace of $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Next, we quote some estimates which will be used in the estimates of $\mathscr{H}$ in Section 3.

Proposition 2.1.[(4, 15)] Suppose that the main assumptions **(M)** hold, and let f be a solution of (1.1) corresponding to initial data f_0. Then we have the following estimates. For $T \geqslant 0$,

(1) $\sup_{p=1}^{n+1} M_p(f(T)) \leqslant C_2 B_* \|f_0\|_{\mathscr{E}}^2$,

(2) $\sup_T \|f(T)\|_{\mathscr{E}} \leqslant (1 + C_3 \eta)\, \|f_0\|_{\mathscr{E}}$,

where C_2 and C_3 are some positive constants independent of time T.

Proposition 2.2.[(15)] Suppose the main assumptions **(M)** in Section 1 hold, and let f and $\bar{f}$ be Bony solutions of (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively. Then $M_{n+1}(f(t), \bar{f}(t))$ is uniformly bounded by the L^1-distance between f_0 and $\bar{f}_0$:

$$M_{n+1}(f(t), \bar{f}(t)) \leqslant C_4 B_* (\|f_0\|_{\mathscr{E}} + \|\bar{f}_0\|_{\mathscr{E}})\, \|f_0 - \bar{f}_0\|_{L^1}.$$

Here C_4 is a positive constant independent of time t.

Remark 2.1. As shown in ref. 15, the above key estimate implies the L^1-stability of Bony solutions.

3. NONLINEAR FUNCTIONALS

In this section, we explicitly construct nonlinear functionals measuring potential interactions and L^1-distance between smooth Bony solutions with compact support in x, and study the time-evolution of these functionals.

3.1. Interaction Potential

In this part, we consider the Lyapunov functional $\mathscr{D}$ measuring potential interactions between particles with different velocities and study its time-variation. Let $f=(f_i)$ and $\bar{f}=(\bar{f}_i)$ be Bony solutions of (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively. For simplicity, we use the following simplified notations for interaction production rates:

$$\Lambda_d(f)(t) \equiv \sum_{1\leqslant i\neq j\leqslant N} \int_{\mathbb{R}^n} (f_i f_j)(x,t)\,dx,$$

$$\Lambda_d(f,\bar{f})(t) \equiv \sum_{1\leqslant i\neq j\leqslant N} \int_{\mathbb{R}^n} (|f_i-\bar{f}_i|\,(f_j+\bar{f}_j))(x,t)\,dx.$$

Moreover, we denote the gain and loss terms in $Q_i(f,f)$ as $Q_i^{\pm}(f,f)$, i.e.,

$$Q_i^+(f,f) \equiv \sum_{(j,k)\in\Lambda_i^+} B_i^{jk} f_j f_k, \qquad Q_i^-(f,f) \equiv \sum_{(j,k)\in\Lambda_i^-} B_i^{jk} f_j f_k.$$

We first rewrite (1.1) as

$$\partial_t(f_i(x+tv_i,t)) = Q_i(f,f)(x+tv_i,t), \tag{3.5}$$

$$\begin{aligned}\partial_t(f_j(x+tv_i+\tau n(v_i,v_j),t)) &= |v_i-v_j|\,\partial_\tau(f(x+tv_i+\tau n(v_i,v_j),t))\\ &\quad + Q_j(f,f)(x+tv_i+\tau n(v_i,v_j),t).\end{aligned} \tag{3.6}$$

For a given time t, we consider i-particles f_i located at $x+tv_i$ and j-particles lying on the half-line $x+tv_i+\tau n(v_i,v_j)$, $\tau\geqslant 0$. Then, after time $\frac{\tau}{|v_i-v_j|}$ elapses, i and j particles will share the same locations:

the new location of f_i particles after time $\dfrac{\tau}{|v_i-v_j|}$ is $x+tv_i+\dfrac{\tau v_i}{|v_i-v_j|}$,

the new location of f_j particles after time $\dfrac{\tau}{|v_i-v_j|}$ is $x+tv_i+\dfrac{\tau(v_i-v_j)}{|v_i-v_j|}$ $+\dfrac{\tau v_j}{|v_i-v_j|}$, hence i and j particles will collide with each other as long as they are admissible pairs in Λ_i, $i=1,..., N$, defined in Section 1. Based on this simple observation, we define an interaction potential $\mathscr{D}$:

$$\mathscr{D}(t) = \sum_{1 \leqslant i \neq j \leqslant N} \mathscr{D}_{ij}(t),$$

$$\mathscr{D}_{ij}(t) \equiv \int_{\mathbb{R}^n} f_i(x+tv_i, t) \left[\int_0^\infty f_j(x+tv_i+\tau n(v_i, v_j), t)\, d\tau \right] dx,$$

where summation is over all pairs (i, j), $i \neq j$, and for Bony solutions, $\mathscr{D}$ is *a priori* bounded by initial data:

$$\begin{aligned}\mathscr{D}(t) &\leqslant \sum_{1 \leqslant i \neq j \leqslant N} \left(\sup_x \int_0^\infty f_j(x+tv_i+\tau n(v_i, v_j), t)\, d\tau \right) \int_{\mathbb{R}^n} f_i(x+tv_i, t)\, dx \\ &\leqslant N(N-1)\, \delta_1(f(t))\, \delta_n(f(t)) \leqslant N(N-1)\, \|f_0\|_{\mathscr{E}}^2 < \infty.\end{aligned}$$

Next we estimate the time-evolution of $\mathscr{D}$.

Lemma 3.1. Suppose that the main assumptions **(M)** in Section 1 hold, and let $f = (f_i)_{i=1}^N$ be a smooth Bony solution of (1.1) with compact support in x corresponding to smooth initial data f_0 satisfying **(M3)**. Then $\mathscr{D}$ satisfies a Lyapunov-type estimate:

$$\mathscr{D}(t) + \frac{v_*}{2} \int_0^t \Lambda_d(s)\, ds \leqslant \mathscr{D}(0).$$

Proof. We only estimate one term $\mathscr{D}_{ij}$. The other terms can be estimated similarly. We use (3.5) and (3.6) to obtain

$$\begin{aligned}&\partial_t (f_i(x+tv_i, t)\, f_j(x+tv_i+\tau n(v_i, v_j), t)) \\ &\quad \leqslant |v_i - v_j|\, \partial_\tau [f_i(x+tv_i, t)\, f_j(x+tv_i+\tau n(v_i, v_j), t)] \\ &\qquad + Q_i^+(f, f)(x+tv_i, t)\, f_j(x+tv_i+\tau n(v_i, v_j), t) \\ &\qquad + f_i(x+tv_i, t)\, Q_j^+(f, f)(x+tv_i+\tau n(v_i, v_j), t). \qquad (3.7)\end{aligned}$$

We integrate (3.7) over $\mathbb{R}^n \times [0, \infty)$ with respect to (x, τ) to get

$$\begin{aligned}\frac{d}{dt}\mathscr{D}_{ij}(t) &\leqslant -\int_{\mathbb{R}^n} |v_i - v_j| \,(f_i f_j)(x+tv_i, t)\, dx \\ &\quad + \int_{\mathbb{R}^n}\int_0^\infty Q_i^+(f, f)(x+tv_i, t)\, f_j(x+tv_i+\tau n(v_i, v_j), t)\, d\tau\, dx \\ &\quad + \int_{\mathbb{R}^n}\int_0^\infty Q_j^+(f, f)(x+tv_i+\tau n(v_i, v_j), t)\, f_i(x+tv_i, t)\, d\tau\, dx \\ &= -\int_{\mathbb{R}^n} |v_i - v_j| \,(f_i f_j)(x+tv_i, t)\, dx + I_{ij}^1(t) + I_{ij}^2(t), \end{aligned} \tag{3.8}$$

where we used the fact that f vanishes at $|x| = \infty$. Next we estimate I_{ij}^1, and the second term I_{ij}^2 can be treated similarly using the change of variables.

$$\begin{aligned} I_{ij}^1(t) &\leqslant \left(\sup_x \int_0^\infty f_j(x+tv_i+\tau n(v_i, v_j), t)\, d\tau \right) \int_{\mathbb{R}^n} Q_i^+(f, f)(x+tv_i, t)\, dx \\ &\leqslant (1+C_3\eta)\, \|f_0\|_{\mathscr{E}} \int_{\mathbb{R}^n} Q_i^+(f, f)(x+tv_i, t)\, dx, \end{aligned} \tag{3.9}$$

where we used the fact that

$$\sup_x \int_0^\infty f_j(x+tv_i+\tau n(v_i, v_j), t)\, d\tau \leqslant \delta_1(f(t)) \leqslant (1+C_3\eta)\, \|f_0\|_{\mathscr{E}}.$$

Now we use the change of variable $(x \to x - \tau n(v_i, v_j))$ and the same estimate as I_{ij}^1 to get

$$I_{ij}^2(t) \leqslant (1+C_3\eta)\, \|f_0\|_{\mathscr{E}} \int_{\mathbb{R}^n} Q_j^+(f, f)(x+tv_i, t)\, dx. \tag{3.10}$$

We combine (3.9) and (3.10) to obtain

$$\begin{aligned}\frac{d\mathscr{D}_{ij}(t)}{dt} &\leqslant -v_* \int_{\mathbb{R}^n} (f_i f_j)(x+tv_i, t)\, dx \\ &\quad + (1+C_3\eta)\, \|f_0\|_{\mathscr{E}} \int_{\mathbb{R}^n} (Q_i^+(f, f) + Q_j^+(f, f))(x+tv_i, t)\, dx. \end{aligned} \tag{3.11}$$

Again in (3.11), we sum up all possible pairs (i, j) with $v_i \neq v_j$ to obtain

$$\frac{d\mathscr{D}(t)}{dt} \leqslant -v_* \Lambda_d(f)(t)$$

$$+(1+C_3\eta)\, \|f_0\|_{\mathscr{E}} \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} (Q_i^+(f, f)+Q_j^+(f, f))(x+tv_i, t)\, dx$$

$$\leqslant -\frac{v_*}{2} \Lambda(f)(t). \tag{3.12}$$

Here we used the fact that

$$B_* N(N-1)\, \|f_0\| \leqslant \eta \ll 1.$$

We integrate (3.12) from $t=0$ to $t=T$ to get

$$\mathscr{D}(T)+\frac{v_*}{2} \int_0^T \Lambda_d(t)\, dt \leqslant \mathscr{D}(0).$$

This completes the proof. ∎

Proof of Theorem 1.1. Let f be a Bony solution corresponding to initial data f_0 satisfying the smallness assumption **M3**. Then by the standard density argument, there exist sequences of smooth C^1 initial data $\{f_0^{(n)}\}_{n=0}^{\infty}$ corresponding to smooth Bony solutions $\{f^{(n)}\}_{n=0}^{\infty}$ with compact support in x such that

$$f_0^{(n)} \to f_0, \qquad f^{(n)} \to f, \qquad \text{as} \quad n \to \infty \text{ in } \mathbf{E}.$$

For more details, refer to ref. 4. By Lemma 3.1, we have

$$\mathscr{D}^{(n)}(t)+\frac{v_*}{2} \int_0^t \Lambda_d^{(n)}(s)\, ds \leqslant \mathscr{D}^{(n)}(0).$$

Here $\mathscr{D}^{(n)}(t)$ and $\Lambda_d^{(n)}(t)$ are the interaction potential and interaction production rates corresponding to $f^{(n)}$. Then it is easy to see that

$$\mathscr{D}^{(n)}(t) \to \mathscr{D}(t), \qquad \int_0^t \Lambda_d^{(n)}(s)\, ds \to \int_0^t \Lambda_d(s)\, ds \qquad \text{as} \quad n \to \infty.$$

Hence we have a Lyapunov-type estimate for a Bony solution f:

$$\mathscr{D}(t)+\frac{v_*}{2}\int_0^t \Lambda_d(s)\,ds \leqslant \mathscr{D}(0).$$

This completes the proof. ∎

Remark 3.1. It follows from the above estimate that

$$\int_0^\infty \int_{\mathbb{R}^n} |Q_i(f,f)|\,(x,t)\,dx\,dt \leqslant \mathscr{D}(0) < \infty.$$

Now we formally set

$$F_{i\infty}(y) \equiv f_{i0}(y)+\int_0^\infty Q_i(f,f)(y+sv_i,s)\,ds;$$

then it is easy to see that

$$\|F_{i\infty}\|_{L^1} \leqslant \|f_{0i}\|_{L^1}+\mathscr{D}(0), \qquad \text{and}$$
$$\|f_i(\cdot+tv_i,t)-F_{i\infty}\|_{L^1} \leqslant \int_t^\infty \int_{\mathbb{R}^n} |Q_i(f,f)|\,(y+sv_i,s)\,dy\,ds \to 0$$
$$\text{as} \quad t\to\infty,$$

and thus $f_i(x,t)\to F_{i\infty}(x-tv_i)$ in $L^1(\mathbb{R}^n)$. Hence, the leading term in the asymptotic response of f_i is a traveling wave. Refer to refs. 4 and 5 for further results on time-asymptotic behavior.

3.2. Nonlinear Functional for L^1-Distance

In this part, we construct a nonlinear functional which is equivalent to the L^1 distance between two Bony solutions of (1.1). As in the previous part, we first construct a nonlinear functional for smooth solutions with compact support in x. Let f and $\bar{f}$ be smooth Bony solutions of (1.1) corresponding to smooth data f_0 and $\bar{f}_0$ satisfying **(M3)** respectively. The equations for the difference $|f_i-\bar{f}_i|$, f_i, and $\bar{f}_i$ are given by:

$$\begin{gathered}\partial_t |f_i-\bar{f}_i|+v_i\cdot\nabla_x |f_i-\bar{f}_i| \leqslant J_i(f,\bar{f}),\\ \partial_t f_j+v_j\cdot\nabla_x f_j = Q_j(f,f), \qquad \partial_t \bar{f}_j+v_j\cdot\nabla_x \bar{f}_j = Q_j(\bar{f},f)\end{gathered} \tag{3.13}$$

where

$$J_i(f, \bar{f}) \equiv \sum_{(j,k) \in \Lambda_i} \frac{|B_i^{jk}|}{2} [|f_j - \bar{f}_j| (f_k + \bar{f}_k) + |f_k - \bar{f}_k| (f_j + \bar{f}_j)].$$

Now we define the nonlinear functional $\mathscr{H}$:

$$\mathscr{L}(t) \equiv \sum_{i=1}^{N} \int_{\mathbb{R}^n} |f_i - \bar{f}_i| (x + tv_i, t)\, dx,$$

$$\mathscr{D}_d(t) \equiv \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} |f_i - \bar{f}_i| (x + tv_i, t) \times \left[\int_0^\infty (f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t)\, d\tau \right] dx,$$

$$\mathscr{H}(t) \equiv \mathscr{L}(t) + K \mathscr{D}_d(t),$$

where K is a large positive constant which will be determined later, and note that when either f or $\bar{f}$ is zero, $\mathscr{D}_d$ reduces to the interaction potential $\mathscr{D}$ defined in Section 3.1.

Lemma 3.2. Suppose the main assumptions **(M)** in Section 1 hold, and let f and $\bar{f}$ be Bony solutions corresponding to initial data f_0 and $\bar{f}_0$ respectively. Then the above subfunctionals satisfy the following estimates:

$$\frac{d\mathscr{L}(t)}{dt} \leqslant B_* N\, \Lambda_d(f, \bar{f})(t),$$

$$\frac{d\mathscr{D}_d(t)}{dt} \leqslant -\frac{v_*}{2} \Lambda_d(f, \bar{f})(t) + \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i| (x + tv_i, t) \cdot [Q_j^+(f, f)(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx.$$

Proof. By the standard density argument, it suffices to show that the above estimates hold for smooth C^1 solutions with compact support in x.

(i) The estimate for $\frac{d\mathscr{L}(t)}{dt}$ follows from (3.13) directly.

(ii) Next we estimate $\frac{d\mathscr{D}_d(t)}{dt}$: As in Lemma 3.1, by direct calculations we have

$$\begin{aligned} &\partial_t(|f_i - \bar{f}_i| (x + tv_i, t)(f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t)) \\ &\quad \leqslant |v_i - v_j|\, \partial_\tau(|f_i - \bar{f}_i| (x + tv_i, t)(f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t)) \\ &\qquad + J_i(f, \bar{f})(x + tv_i, t)(f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t) \\ &\qquad + |f_i - \bar{f}_i| (x + tv_i, t)(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t). \end{aligned} \tag{3.14}$$

We integrate $\sum_{1 \leqslant i \neq j \leqslant N}$ (3.14) over $\mathbb{R}^n \times [0, \infty)$ in (x, τ) to get

$$\begin{aligned}
\frac{d\mathscr{D}_d(t)}{dt} &\leqslant -v_* \Lambda_d(f, \bar{f})(t) \\
&\quad + B_* N(N-1)\left[\sup_x \int_0^\infty (f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t)\, d\tau\right] \Lambda_d(f, \bar{f}) \\
&\quad + \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i|\,(x + v_i t, t) \\
&\quad \cdot (Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx \\
&\leqslant -\frac{v_*}{2} \Lambda_d(f, \bar{f}) + \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i|\,(x + tv_i, t) \\
&\quad \cdot (Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx,
\end{aligned}$$

where we used

$$\sup_x \int_0^\infty (f_j + \bar{f}_j)(x + tv_i + \tau n(v_i, v_j), t)\, d\tau \leqslant (1 + C_3 \eta)\, \|f_0\|_{\mathscr{E}}.$$

This completes the proof. ∎

Proof of Theorem 1.2. By the definition of $\mathscr{H}$ and Lemma 3.2, we have

$$\begin{aligned}
\frac{d\mathscr{H}(t)}{dt} &\leqslant \left(B_* N - \frac{K v_*}{2}\right) \Lambda_d(f, \bar{f})(t) + K \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i|\,(x + tv_i, t) \\
&\quad \cdot [(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx.
\end{aligned}$$

We choose K large enough so that

$$B_* N - \frac{K v_*}{2} < 0.$$

Then for such K, we have

$$\begin{aligned}
&\frac{d\mathscr{H}(t)}{dt} + C_1 \Lambda_d(f, \bar{f})(t) \\
&\quad \leqslant K \sum_{1 \leqslant i \neq j \leqslant N} \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i|\,(x + v_i t, t) \\
&\qquad \cdot [(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx,
\end{aligned}$$

where C_1 is a positive constant and the summation is over all pairs (i, j) with $i \neq j$. We integrate the above inequality from $t = 0$ to $t = T$ to get

$$\begin{aligned}
&\mathscr{H}(T) + C_1 \int_0^T \Lambda_d(f, \bar{f})(t)\, dt \\
&\quad \leqslant \mathscr{H}(0) + K \sum_{1 \leqslant i \neq j \leqslant N} \int_0^T \int_{\mathbb{R}^n} \int_0^\infty |f_i - \bar{f}_i|\,(x + tv_i, t) \\
&\qquad \cdot [(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(x + tv_i + \tau n(v_i, v_j), t)]\, d\tau\, dx\, dt. \qquad (3.15)
\end{aligned}$$

We denote the second term of the right-hand side of (3.15) by $S(t)$.

We claim:

$$S(t) = \mathcal{O}(\eta^2)\, \|f_0 - \bar{f}_0\|_{L^1}.$$

For a point $P \in \Pi \equiv \mathbb{R}^n \times \mathbb{R}$, we set

$N_i(P) \equiv$ the intersection point between a backward characteristic line with velocity $-V_i = (-v_i, -1)$ issued from P and a hyperplane $\mathbb{R}^n \times \{s = 0\}$,

$N_i(P)\, P \equiv$ the section of a characteristic line with velocity V_i connecting $N_i(P)$ and P,

$\hat{l}(N, V_i, t) \equiv$ the section of a characteristic line issued from N with velocity V_i in the time zone $\mathbb{R}^n \times [0, t]$,

$$l_{ij}(P, \tau) \equiv P + \tau n(v_i, v_j), \quad \tau \geqslant 0, \qquad l_{ij}^+(P) \equiv \{l_{ij}(P, \tau) \colon \tau \geqslant 0\}.$$

In the sequel, we sometimes drop the P-dependence in $N_i(P)$, i.e.,

$$N_i(P) = N_i.$$

We integrate (3.13) along the characteristic line $N_i P$ to get

$$|f_i - \bar{f}_i|\,(P) \leqslant |f_{0i} - \bar{f}_{0i}|\,(N_i) + \int_{R_i \in N_i P} J_i(f, \bar{f})(R_i)\, dR_i. \qquad (3.16)$$

We substitute (3.16) into $S(t)$ to get

$$S(t) \equiv \sum_{1 \leqslant i \neq j \leqslant N} \int_{P \in \Pi \cap (\mathbb{R}^n \times [0,t])} |f_i - \bar{f}_i| (P) \times \left[\int_0^\infty Q_j^+(f, f)(l_{ij}(P, \tau))\, d\tau \right] d^{n+1}P$$

$$\leqslant \sum_{1 \leqslant i \neq j \leqslant N} \int_{P \in \Pi \cap (\mathbb{R}^n \times [0,t])} \int_0^\infty |f_{0i} - \bar{f}_{0i}| (N_i)(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f})) \times (l_{ij}(P, \tau))\, d\tau\, d^{n+1}P$$

$$+ \sum_{1 \leqslant i \neq j \leqslant N} \int_{P \in \Pi \cap (\mathbb{R}^n \times [0,t])} \int_{R_i \in N_i P} \int_0^\infty J_i(f, \bar{f})(R_i)(Q_j^+(f, f) + Q_j^+(\bar{f}, \bar{f}))(l_{ij}(P, \tau))\, d\tau\, dR_i\, d^{n+1}P. \tag{3.17}$$

In order to estimate (3.17), it suffices to consider the following two types of space-time integrals:

$$II_{ij}^1(t) \equiv \int_{P \in \Pi \cap (\mathbb{R}^n \times [0,t])} |f_{0i} - \bar{f}_{0i}| (N_i) \left[\int_0^\infty Q_j^+(f, f)(l_{ij}(P, \tau))\, d\tau \right] d^{n+1}P$$

$$II_{ij}^2(t) \equiv \int_{P \in \Pi \cap (\mathbb{R}^n \times [0,t])} \int_{R_i \in N_i P} J_i(f, \bar{f})(R_i) \times \left[\int_0^\infty Q_j^+(f, f)(l_{ij}(P, \tau))\, d\tau \right] dR_i\, d^{n+1}P.$$

Before we estimate $II_{ij}^1(t)$ and $II_{ij}^2(t)$, we note that for a fixed point $N \in \mathbb{R}^n$ and t, the plane $\bar{\Pi}(t)$ defined by

$$\bar{\Pi}(t) =: \{\hat{R}_i + l_{ij}(\hat{R}_i, \tau) \mid \hat{R}_i \in \hat{l}(N, V_i, t), \tau \geqslant 0\}$$

is a part of the characteristic 2-plane spanned by V_i and V_j, i.e.,

Claim. V_i, and V_j are parallel to $\bar{\Pi}(t)$.

Proof of the Claim. We first notice that the vectors $\hat{R}_i$ and $l_{ij}(\hat{R}_i, \tau)$ can be written as follows:

$$\hat{R}_i = N + kV_i, \qquad l_{ij}(\hat{R}_i, \tau) = \hat{R}_i + \frac{\tau(V_j - V_i)}{|V_j - V_i|},$$

$$\text{for some constants} \quad k, \tau \geqslant 0.$$

This yields

$$\hat{R}_i + l_{ij}(\hat{R}_i, \tau) = N + \left(k - \frac{\tau}{|V_j - V_i|}\right) V_i + \left(\frac{\tau}{|V_j - V_i|}\right) V_j.$$

Hence characteristic vectors V_i and V_j are parallel to $\bar{\Pi}(t)$.

Next we estimate $II^1_{ij}(t)$ and $II^2_{ij}(t)$ as follows:

$$\begin{aligned} II^1_{ij}(t) &= \int_{N \in \mathbb{R}^n} \int_{\hat{R}_i \in \hat{l}(N, V_i, t)} |f_{0i} - \bar{f}_{0i}| \, (N_i(\hat{R}_i)) \left[\int_0^\infty Q_j^+(f, f)(l_{ij}(\hat{R}_i, \tau)) \, d\tau \right] \\ &\quad \times d\hat{R}_i \, d^n N \\ &= \int_{N \in \mathbb{R}^n} |f_{0i} - \bar{f}_{0i}| \, (N) \left[\int_{\hat{R}_i \in \hat{l}(N, V_i, t)} \int_0^\infty Q_j^+(f, f)(l_{ij}(\hat{R}_i, \tau)) \, d\tau \, d\hat{R}_i \right] \\ &\quad \times d^n N \\ &\leqslant N_*^+ M_2(f(t)) \, \|f_{0i} - \bar{f}_{0i}\|_{L^1}, \end{aligned}$$

where N_* is the maximal size of Λ_i defined in Section 1 and we used

$$N_i(\hat{R}_i) = N, \qquad \hat{R}_i \in \hat{l}(N, V_i, t).$$

Similarly, we have

$$\begin{aligned} II^2_{ij}(t) &= \int_{N \in \mathbb{R}^n} \int_{\hat{R}_i \in \hat{l}(N, V_i, t)} \int_{R_i \in N\hat{R}_i} J_i(f, \bar{f})(R_i) \\ &\quad \times \left(\int_0^\infty Q_j^+(f, f)(l_{ij}(\hat{R}_i, \tau)) \, d\tau \right) dR_i \, d\hat{R}_i \, d^n N \\ &\leqslant \int_{N \in \mathbb{R}^n} \left[\int_{R_i \in \hat{l}(N, V_i, t)} J_i(f, \bar{f})(R_i) \, dR_i \right] \\ &\quad \times \left[\int_{\hat{R}_i \in \hat{l}(N, V_i, t)} \int_0^\infty Q_j^+(f, f)(l_{ij}(\hat{R}_i, \tau)) \, d\tau \, d\hat{R}_i \right] d^n N \\ &\leqslant N_* N_*^+ M_2(f(t)) \, M_{n+1}(f(t), \bar{f}(t)). \end{aligned}$$

Here we used

$$\int_{R_i \in N\hat{R}_i} J_i(f, \bar{f})(R_i) \, dR_i \leqslant \int_{\hat{R}_i \in \hat{l}(N, V_i, t)} J_i(f, \bar{f}) \, d\hat{R}_i \qquad \text{for} \quad \hat{R}_i \in \hat{l}(N, V_i, t).$$

In (3.17), we use Propositions 2.1 and 2.2 to obtain

$$\begin{aligned}S(t) &\leqslant (N-1)\, N_*^+(M_2(f(t))+M_2(\bar{f}(t)))\, \|f_0-\bar{f}_0\|_{L^1}\\&\quad +N(N-1)\, N_* N_*^+(M_2(f(t))+\bar{M}_2(\bar{f}(t)))\, M_{n+1}(f(t),\bar{f}(t))\\&\leqslant C_2(N-1)\, N_*^+ B_*(\|f_0\|_{\mathscr{E}}^2+\|\bar{f}_0\|_{\mathscr{E}}^2)\, \|f_0-\bar{f}_0\|_{L^1}\\&\quad +C_2C_4N(N-1)\, N_*N_*^+B_*^2(\|f_0\|_{\mathscr{E}}^2+\|\bar{f}_0\|_{\mathscr{E}}^2)(\|f_0\|_{\mathscr{E}}+\|\bar{f}_0\|_{\mathscr{E}})\, \|f_0-\bar{f}_0\|_{L^1}\\&=\mathcal{O}(\eta^2)\, \|f_0-\bar{f}_0\|_{L^1}.\end{aligned}$$

We combine the above estimates to obtain

$$\mathscr{H}(t)+C_1\int_0^t \Lambda_d(f,\bar{f})(s)\, ds \leqslant \mathscr{H}(0)+\mathcal{O}(\eta^2)\, \mathscr{L}(0)\leqslant \bar{C}_1\mathscr{H}(0).$$

where C_1 and $\bar{C}_1$ are positive constants independent of t and we used $\mathscr{L}(0)\leqslant\mathscr{H}(0)$. This completes the proof. ∎

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